The Simplex Method: Conceptual Guide

Linear algebra and optimization step-by-step

Generated by Simplex Solver Pro

1 1. Standard Form and Mathematical Basis

Every Linear Programming problem must be converted into standard form before applying the algorithm. Mathematically, we seek to optimize $\max Z = c^T x$ subject to $Ax = b$ with $x \geq 0$.

- $\leq$ **Constraints:** Add a *slack variable* ($+s_i$). It represents unused resources.
- $\geq$ **Constraints:** Subtract a *surplus variable* ($-e_i$). It represents the excess over the minimum requirement.
- $=$ **Constraints:** Introduce an *artificial variable* ($+a_i$) exclusively to force an initial identity matrix and find a Basic Feasible Solution (BFS).

2 2. The Base Algorithm: Pivot Criteria

The Simplex method moves from vertex to vertex on the solution polyhedron, improving the objective function. Each tableau represents a BFS where variables are divided into *Basic* (form the basis B) and *Non-Basic* (value of zero).

A. Entering Criterion (Which variable improves the model?)

We evaluate the reduced costs, represented by the $Z_j - C_j$ row. This value indicates the net change in the objective function per unit of the variable entering the basis.

- **Maximization:** The variable with the **most negative** $Z_j - C_j$ enters. The iteration stops (optimal) when all $Z_j - C_j \geq 0$.
- **Minimization:** The variable with the **most positive** $Z_j - C_j$ enters. The optimum is reached when all $Z_j - C_j \leq 0$.

B. Leaving Criterion (Minimum Ratio Test)

Ensures the new solution does not leave the feasible region (non-negativity condition).

$$\theta = \min \left\{ \frac{\text{Right Hand Side (RHS)}}{\text{Pivot Column Coefficient}} \;\middle|\; \text{Coefficient} > 0 \right\}$$

The variable in the winning row leaves the basis. If all coefficients are ≤ 0, the problem is **Unbounded**.

3 3. Initialization Methods (Artificial Variables)

When the original problem does not provide a trivial identity basis, we must force one.

- **Big M Method:** Penalizes artificial variables in the objective function with a massive cost ($-M$ for max, $+M$ for min). The algorithm will quickly drive them out of the basis to avoid the penalty.
- **Two-Phase Method:**
  - *Phase I:* Minimizes exclusively the sum of the artificial variables ($W = \sum a_i$). If the minimum is 0, we have a valid BFS and the artificials disappear. If it is > 0, the problem is **Infeasible**.
  - *Phase II:* Restores the original objective function and optimizes starting from the basis found in Phase I.

4 4. Special Cases (Post-Optimal Analysis and Geometry)

- **Degeneracy:** Occurs when there is a tie in the minimum ratio test. A basic variable takes the value 0. Geometrically, this means redundant constraints meet at the same vertex. It can cause infinite cycling if Bland's rule is not applied.
- **Multiple Optima:** In the optimal tableau, a *non-basic* variable has a reduced cost of $Z_j - C_j = 0$. Entering it into the basis will change the variable mix without altering the optimal Z value. The objective function is parallel to a constraint.
- **Infeasibility:** At the optimal Big M tableau (or end of Phase I), an artificial variable remains in the basis with a value > 0. The polyhedron is empty.

Need to check your tableaus or analyze a degenerate case?
Input your matrix and visualize the iterations step-by-step at:
simplexsolverpro.com/en